Solutions

Closed book, closed notes, calculators OK. The Signal Processing problem consists of two parts, each comprising multiple questions, points as marked (point sum = 40 = perfect score); nominal duration = 1 hour.

Part A: Filtering, (down-)sampling, and the roots of unity [sum of points = 25]

- [**3 points**] For $a \in \mathbb{C}$, define $s(N) := 1 + a + a^2 + \cdots + a^{N-1}$. Express $s(N+1)$ as a function of $s(N)$ *in two different ways*, and use the resulting expressions to derive a closed-form expression for $s(N)$.

$$s(N+1) = s(N) + a^N = 1 + as(N) \Rightarrow (1-a)s(N) = 1 - a^N \Rightarrow s(N) = \frac{1-a^N}{1-a}.$$

- [**5 points**] Consider a discrete-time filter with impulse response

$$h(k) = \begin{cases} 1, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

where k is the discrete time variable, and N is a non-negative integer constant (the length of the impulse response). Compute the discrete-time Fourier transform $H(e^{j\omega}) := \sum_{k=-\infty}^{+\infty} h(k)e^{-j\omega k}$.

$$H(e^{j\omega}) = \sum_{k=0}^{N-1} e^{-j\omega k} = \sum_{k=0}^{N-1} (e^{-j\omega})^k = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}.$$

- [**5 points**] Suppose that a discrete-time signal $x(k)$ is input to the above filter, and the output $y(k) = 0$, $\forall k$. What can you say about (i.e., can you *characterize*) the input signal $x(k)$ in this case? Be as specific as you can.

Notice that $H(e^{j\omega})$ is zero at all ω for which $e^{-j\omega N} = 1$ (and $e^{-j\omega} \neq 1$), i.e., at $\omega N = \ell 2\pi \Leftrightarrow \omega = \frac{2\pi}{N}\ell$, $\ell \in \{1, \cdots, N-1\}$ ($\pm$ integer multiples of 2π). At $\omega = 0$, on the other hand, $H(e^{j0}) = \sum_{k=0}^{N-1} e^{-j0} = N$. Thus the filter suppresses all frequency content at $\omega = \frac{2\pi}{N}\ell$, $\ell \in \{1, \cdots, N-1\}$ and passes all other frequencies. When $y(k) = 0$, $\forall k$, what you can say is that $x(k)$ is confined to have content only at $\omega = \frac{2\pi}{N}\ell$, $\ell \in \{1, \cdots, N-1\}$. In particular, $x(k)$ need not be the all-zero signal.

- [**5 points**] Now assume that the discrete-time Fourier transform $X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$ of $x(k)$ satisfies $X(e^{j\omega}) = 0$, $\forall |\omega| > \omega_c$. Find a condition on ω_c that allows recovery of the input signal $\{x(k)\}_{k=-\infty}^{\infty}$ from the output signal $\{y(k)\}_{k=-\infty}^{\infty}$.

The first zero of $H(e^{j\omega})$ occurs at frequency $\omega = \frac{2\pi}{N}$, so if $\omega_c < \frac{2\pi}{N}$, then recovery of the input signal is theoretically possible via 'inverse filtering' by $\frac{1}{H(e^{j\omega})}$ over the sub-band of interest.

- **[2 points]** Suppose you down-sample the filter's output $\{y(k)\}_{k=-\infty}^{\infty}$ by a factor of N, and let $\{v(k)\}_{k=-\infty}^{\infty}$ with $v(k) := y(Nk)$, $\forall k \in \mathbb{Z}$, be the down-sampled signal. Under what condition can you recover the signal $\{y(k)\}_{k=-\infty}^{\infty}$ from the signal $\{v(k)\}_{k=-\infty}^{\infty}$? (no need to prove the (down-)sampling theorem here, simply state the condition).

  The condition is that $\{y(k)\}_{k=-\infty}^{\infty}$ should be bandlimited to within $\omega_c < \frac{\pi}{N}$. This will be true if and only if $\{x(k)\}_{k=-\infty}^{\infty}$ is bandlimited to within $\omega_c < \frac{\pi}{N}$.

- **[5 points]** Write out an expression for $v(k)$ as a function of the filter's *input* signal. Combining the above results, state a condition under which you can exactly recover the filter's input signal $\{x(k)\}_{k=-\infty}^{\infty}$, from the down-sampled version $\{v(k)\}_{k=-\infty}^{\infty}$ of the filter's output signal. Comment on the result.

  $v(k) = \sum_{m=1}^{N} x((k-1)N + m)$. The condition is simply that $\{x(k)\}_{k=-\infty}^{\infty}$ should be bandlimited to within $\omega_c < \frac{\pi}{N}$. Under this condition, $\{x(k)\}_{k=-\infty}^{\infty}$ can be *perfectly recovered from its partial non-overlapping sums.* This speaks for the 'severity' of the bandlimited assumption, which is often taken 'lightly'.

Part B: Phase retrieval? [sum of points = 15]

- **[5 points]** Show by simple counter-example that it is generally impossible to recover a discrete-time signal $\{x(k)\}_{k=-\infty}^{\infty}$ from only the magnitude $|X(e^{j\omega})|$ of its discrete-time Fourier transform $X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$. I.e., show that we need the phase of $X(e^{j\omega})$ too, $|X(e^{j\omega})|$ is not enough.

  A simple counter-example is that $x(k)$ and $x(k-\ell)$ have the same Fourier transform magnitude, so you cannot recover the 'correct' shift. E.g., $\delta(k) \leftrightarrow 1$ and $\delta(k-\ell) \leftrightarrow e^{-j\omega\ell}$.

- **[10 points]** Describe how you can construct a non-trivial class of signals $\{x(k)\}_{k=-\infty}^{\infty}$ for which perfect recovery of $\{x(k)\}_{k=-\infty}^{\infty}$ is possible from $|X(e^{j\omega})|$ only. Hint: How can you generate $X(e^{j\omega}) \geq 0$ (non-negative real) $\forall\omega$?

  Let $x(k)$ be the convolution of $h(k)$ and $g(k) := h^*(-k)$, where * stands for complex conjugate. Then

  $$G(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h^*(-k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h^*(k)e^{j\omega k} = \left(\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right)^* = H^*(e^{j\omega}),$$

  and since convolution in time corresponds to multiplication in frequency,

  $$X(e^{j\omega}) = H(e^{j\omega})G(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2 \geq 0, \ \forall\omega.$$

  So we only need to take the inverse Fourier transform of $|X(e^{j\omega})|$, which is equal to the inverse Fourier transform of $X(e^{j\omega})$.

  Remark: This is in fact the only way to generate $X(e^{j\omega}) \geq 0$ (non-negative real) $\forall\omega$. The only if part is the Fejér-Riesz spectral factorization theorem, but I don't ask you to prove this here.